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ORIGINAL PAPER

Integral of motion and nonlinear dynamics of three Duffing oscillators with weak or strong bidirectional coupling

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Abstract In this work, we present a system composed of three identical Duffing oscillators coupled bidirectionally. Starting from a Lagrangian that describes the system, an integral of motion is obtained by means of Noether's theorem. The dynamics of the model is studied using bifurcation diagrams, Lyapunov exponents, time-series analysis, phase spaces, Poincaré sections, spatiotemporal and integral of motion planes. The analysis focuses on the monostable and bistable cases of the Duffing oscillator potential, in which a confined movement is guaranteed. In particular, it is observed that the system shows a chaotic behavior for small values of the coupling parameter for the bistable case. This is one of the first articles in the literature in which non-trivial integrals of motion are obtained for a system of three Duffing oscillators coupled bidirectionally. It is worth pointing out that there are some reports in the literature on integrals of motion for unidirectionally coupled nonlinear Duffing oscillators, but the study carried out in this work for bidirectionally coupled systems with more than two nonlinear Duffing oscillators is certainly one of the first.

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1 Introduction

The Duffing oscillator (DO) is one of the most important nonlinear models in mathematical physics, mechanics and engineering, among other disciplines. The DO was originally proposed by Georg Duffing to model the motion of a single degree of freedom mechanical system with harmonic excitation and a nonlinear restoring force [19]. Moreover, a generalization of this nonlinear oscillator is found in the form of the Emden–Fowler equation [70]. Given its characteristic oscillation and chaotic nature, the DO has a wide application in different areas of mathematical physics (see [27,37] and references therein). In particular, the DO has been employed in the investigation of dynamical systems and their bistability [1,22], bifurcation problems [55] and control theory [53]. More specifically, in terms of coupled systems, the DO has been studied in the investigation of coupled chaotic oscillators with harmonic excitation [43] and the analysis of the synchronization behavior [8,60].

According to Noether's theorem, it is clear that an integral of motion can give information about the symmetry properties of a given system, and it is even possible to obtain stability properties of the solutions without the need to solve equations explicitly [17,40,41]. In fact, the integrals of motion have potential applications in nonlinear systems or the study of diseases, such as tuberculosis or dengue [26]. Moreover, since the discovery of the Ermakov invariant [20,42,46], the search for integrals of motion has been a common problem in systems that depend explicitly on time. The physical interpretation of these integrals is more complex and is associated with a space-time transformation of the coordinates of a system, resulting in a transformation of the energy as discussed in [23,54]. The works published by Ray and Reid show generalizations of the Ermakov invariant [58,59], and it is also possible to construct integrals of motion in stochastic systems [12–14,66]. Even in the current topic of investigation in the present manuscript, some works have been published on integrals of motion for the DO with variable coefficients [39,68], stochastic noise [67] or higher-order nonlinearities [24,48]. The techniques to find the integrals of motion are essentially based on algebraic arguments [33,35], Noether's theorem [25,47] or methods based on transformation groups [11].

Without any doubt, coupled systems are one of the cornerstones of control theory [15,16,57]. Various applications can be found in areas as diverse as electronics [18], communications [71], robotics [28,61], biology [52], biophysics [63], etc. In the case of coupled DOs, several works can be found on resonant effects [31], exact solutions [44], vibration analysis [45], motif networks [5,6,29], route to chaos and bifurcation analysis [6,38,50].

In particular, the investigation of ring-coupling systems has gained importance since Alan Turing's seminal article on morphogenesis [65], in view that this configuration favors phase propagation [21,69]. In that respect, the work by Keener [34] deserves special merit. In that article, it is shown that propagation fails precisely when the coupling is weak. Various consequences are discussed thoroughly therein, especially in the context of cardiophysiology. Different studies about rings with both unidirectional and bidirectional coupling have been carried out with different types of oscillators in their nodes [2,32,62,72]. In the particular case of the Duffing oscillator, there are many works focused primarily on unidirectional coupling [5,6,9,10,29,30,56,64], even a report on a fractional extension of this oscillator [4] and a study on rotating waves [3] have been recently published. However, information on the dynamics of bidirectional ring-coupled multiple Duffing oscillators is scarce. In view of these facts, the aim of this work is to contribute in the investigation of those complex systems.

On the other hand, to the best of our knowledge, except for two nonlinear weakly coupled oscillators [36] and two bidirectionally coupled Duffing oscillators with constant coefficients [51], there are no articles in the literature on integrals of motion for three Duffing oscillators coupled bidirectionally. In the present work, we will consider a system of three Duffing oscillators strongly and bidirectionally coupled. From the practical point of view, this configuration (also known as bidirectional three-node motif) plays a fundamental role in complex networks analysis [7,49]. That is why the main objective of this article is to propose an integral of motion for this system of coupled oscillators.

The present manuscript is organized as follows. In Sect. 2, we recall the basic elements to obtain an integral of motion in the context of Noether's theorem, including the Lagrangian and the Euler–Lagrange equations. In Sect. 3, the model and Lagrangian for the DO coupled bidirectionally are proposed, and the corresponding integral of motion is calculated. Dynamics of the nonlinear system is analyzed by means of spatiotemporal and integral of motion planes, bifurcation diagrams, Lyapunov exponents, time series, phase spaces and Poincaré sections in Sect. 4. Finally, we close the work with a brief conclusion and a summary of the most important results obtained.

2 Preliminaries

The present section is devoted to review the formalism based on Noether's theorem to obtain integrals of motion for a system with several dependent variables. This formalism can be consulted in [25]. The starting point is to propose a group transformation of the form

$$X = \xi(x_i, t)\frac{\partial}{\partial t} + \sum_j \eta_j(x_i, t)\frac{\partial}{\partial x_j}. \tag{1}$$

If the symmetry transformation associated with this operator keeps the action $S = \int L(x_i, \dot{x}_i, t)\mathrm{d}t$ invariant, then the integral of motion corresponding to the system described by a given Lagrangian $L(x_i, \dot{x}_i, t)$ can be obtained.

Suppose next that the space–time variation of the action is equal to zero. More precisely, assume that

$$\begin{aligned} \delta S &= \delta \int_{t_1}^{t_2} L(x_i, \dot{x}_i, t)\mathrm{d}t \\ &= \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial x_i}h_i + \frac{\partial L}{\partial \dot{x}_i}\dot{h}_i\right)\mathrm{d}t + L\delta t\Bigg|_{t_1}^{t_2} = 0, \end{aligned} \tag{2}$$

where $h_i = \delta x_i - \dot{x}_i\delta t$. After integrating by parts, we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} \left(\sum_i \left[\frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\right]h_i\right)\mathrm{d}t \\ &+ \left(\sum_i \frac{\partial L}{\partial \dot{x}_i}h_i + L\delta t\right)\Bigg|_{t_1}^{t_2} = 0. \end{aligned} \tag{3}$$

Each of the terms of the first summation cancels out since L satisfies the Euler–Lagrange equations. Meanwhile, the remaining terms of Eq. (3) give rise to the integral of motion associated with the transformation.

If the variation performed is based on the group transformation (1), then $\delta t = \epsilon\xi$ and $\delta x_i = \epsilon\eta_i$, where ϵ is infinitesimal. Therefore, the integral of motion is obtained as

$$I = \sum_i \left[(\xi\dot{x}_i - \eta_i)\frac{\partial L}{\partial \dot{x}_i}\right] - \xi L + F, \tag{4}$$

where the function $F = F(x_i, t)$ is derived from the total derivative added to the non-transformed Lagrangian. Differentiating (4) with respect to time and equating to zero, we arrive at the following condition:

$$\begin{aligned} \xi\frac{\partial L}{\partial t} &+ \dot{\xi}L + \sum_i \left[\eta_i\frac{\partial L}{\partial x_i} + \left(\dot{\eta}_i - \dot{x}_i\dot{\xi}\right)\frac{\partial L}{\partial \dot{x}_i}\right] \\ &= \dot{F}(x_i, t). \end{aligned} \tag{5}$$

Here, $\dot{\xi}$, $\dot{\eta}_i$ and $\dot{F}$ can be written explicitly as

$$\begin{aligned} \dot{\xi} &= \frac{\partial \xi}{\partial t} + \sum_i \frac{\partial \xi}{\partial x_i}\dot{x}_i, \\ \dot{\eta}_i &= \frac{\partial \eta_i}{\partial t} + \sum_j \frac{\partial \eta_i}{\partial x_j}\dot{x}_j, \\ \dot{F} &= \frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial x_i}\dot{x}_i. \end{aligned} \tag{6}$$

Therefore, if a known Lagrangian L is substituted in condition (5), then the functions ξ, η_i and F are obtained. When we substitute those expressions into Eq. (4), we arrive at the desired integral of motion.

3 Integral of motion

Consider a system of three identical DOs coupled bidirectionally. Precisely, assume that $x_1, x_2, x_3 : [0, \infty) \rightarrow \mathbb{R}$ have continuous derivatives up to the second order, which they satisfy the system of ordinary differential equations

$$\begin{aligned} &\ddot{x}_1 + \Omega^2 x_1 + \alpha x_1^3 + \beta(x_1 - x_2) + \beta(x_1 - x_3) = 0, \\ &\ddot{x}_2 + \Omega^2 x_2 + \alpha x_2^3 + \beta(x_2 - x_1) + \beta(x_2 - x_3) = 0, \\ &\ddot{x}_3 + \Omega^2 x_3 + \alpha x_3^3 + \beta(x_3 - x_1) + \beta(x_3 - x_2) = 0, \end{aligned} \tag{7}$$

where Ω^2, α and the coupling parameter β are real constants. A suitable Lagrangian describing the nonlinear system (7) is given by

$$\begin{aligned} L = &\sum_{i=1}^{3}\left[\frac{1}{2}\dot{x}_i^2 - \frac{1}{2}\left(2\beta + \Omega^2\right)x_i^2 - \frac{\alpha}{4}x_i^4\right] \\ &+ \beta(x_1x_2 + x_1x_3 + x_2x_3). \end{aligned} \tag{8}$$

It is easy to show that if we substitute this Lagrangian into the Euler–Lagrange equations, then we obtain the original system (7). Using the methodology described in the previous section, we will obtain now an integral of motion corresponding to system (7) using the proposed Lagrangian. First, substitute the Lagrangian (8) into condition (5) taking into account the explicit

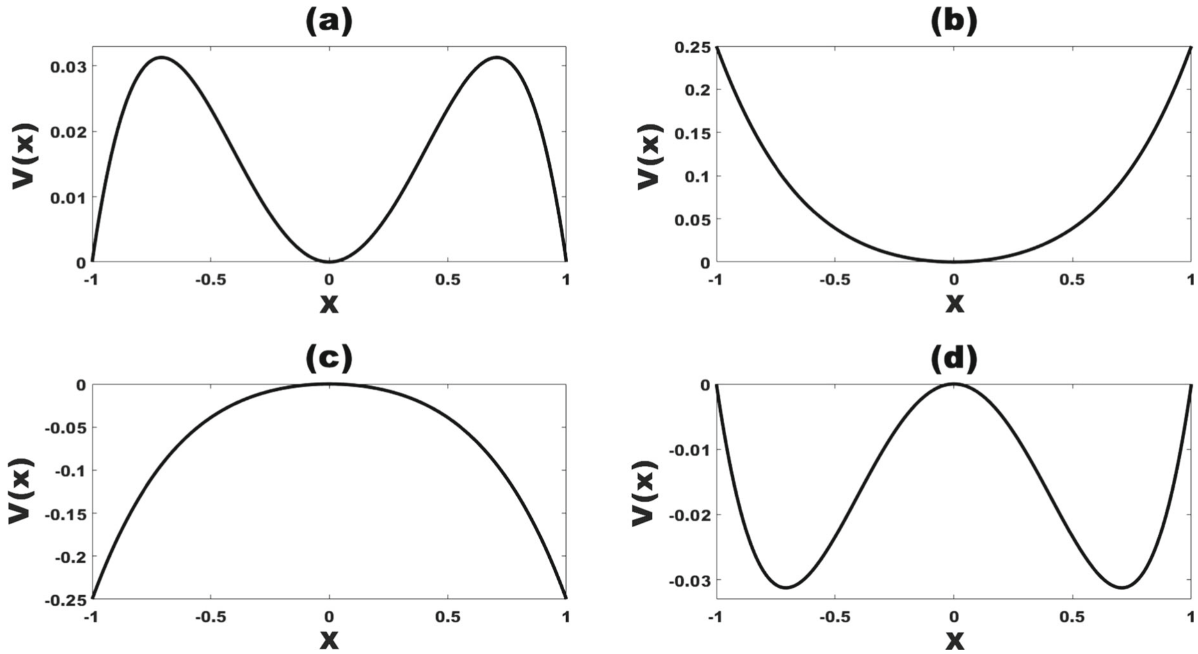

Fig. 1 Potential wells corresponding to an uncoupled Duffing oscillator as functions of their parameters, where **a** $\Omega^2 = 0.25$ and $\alpha = -0.5$, **b** $\Omega^2 = 0.25$ and $\alpha = 0.5$, **c** $\Omega^2 = -0.25$ and $\alpha = -0.5$ and **d** $\Omega^2 = -0.25$ and $\alpha = 0.5$

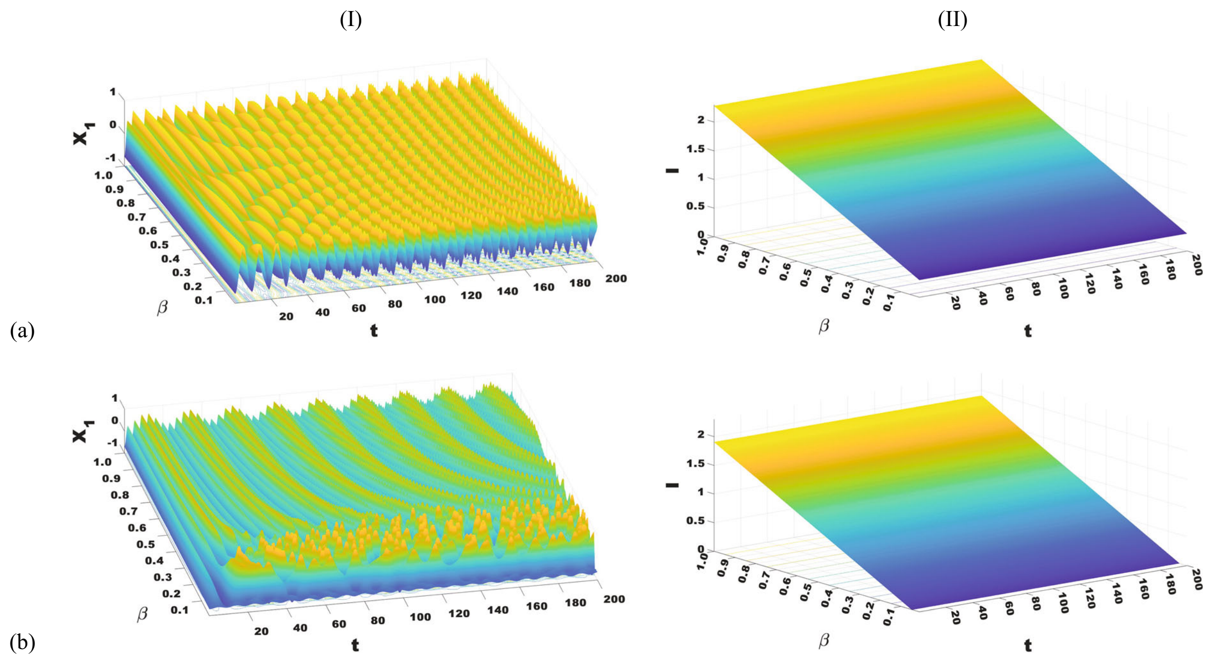

Fig. 2 (I) Spatiotemporal and (II) integral of motion planes for **a** single-well case and **b** double-well case

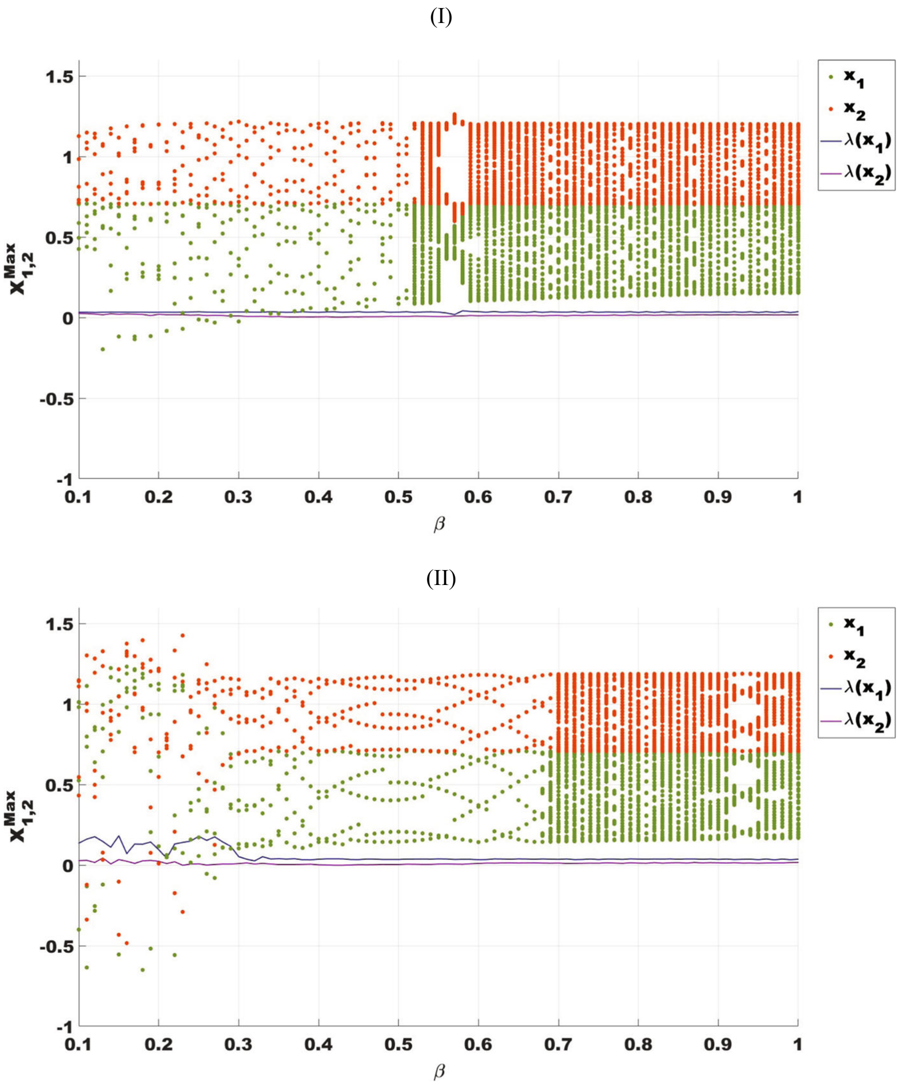

Fig. 3 Local maxima (green, red) and Lyapunov exponents (blue, magenta) of x_1 and x_2 for (I) single-well case and (II) double-well case

expressions (6). In such way, we obtain the identity

$$\begin{aligned}\frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial x_i}\dot{x}_i &= \sum_i \Bigg[-\frac{1}{2}\dot{\xi}\dot{x}_i^2 - \frac{\alpha}{4}\dot{\xi}x_i^4 - \alpha\eta_i x_i^3 - \Omega^2\eta_i x_i \\ &\quad - \frac{1}{2}\left(2\beta + \Omega^2\right)\dot{\xi}x_i^2 + \left(\frac{\partial \eta_i}{\partial t} + \sum_j \frac{\partial \eta_i}{\partial x_j}\dot{x}_j\right)\dot{x}_i\Bigg] \\ &\quad + \beta\dot{\xi}\left(x_1x_2 + x_1x_3 + x_2x_3\right),\end{aligned} \tag{9}$$

where $\dot{\xi}$ is written explicitly using (6). Taking into account the linear independence of the coordinates and their respective velocities, it is easy to show that $\eta_i = 0$ and the functions ξ and F must be arbitrary constants.

As conclusion, if we substitute these results into (4), setting $\xi = 1$ and $F = 0$, then the integral of motion is

$$\begin{aligned}I = \sum_i \left[\frac{1}{2}\dot{x}_i{}^2 + \frac{1}{2}\left(2\beta + \Omega^2\right)x_i^2 + \frac{\alpha}{4}x_i^4\right] \\ - \beta\left(x_1x_2 + x_1x_3 + x_2x_3\right).\end{aligned} \tag{10}$$

It can be checked by direct differentiation that (10) is an integral of motion. In the following section, we will show numerical results of the dynamic behavior of the coupled system (7), for different cases of the potential form of the DO, using time series, bifurcation diagrams and phase spaces.

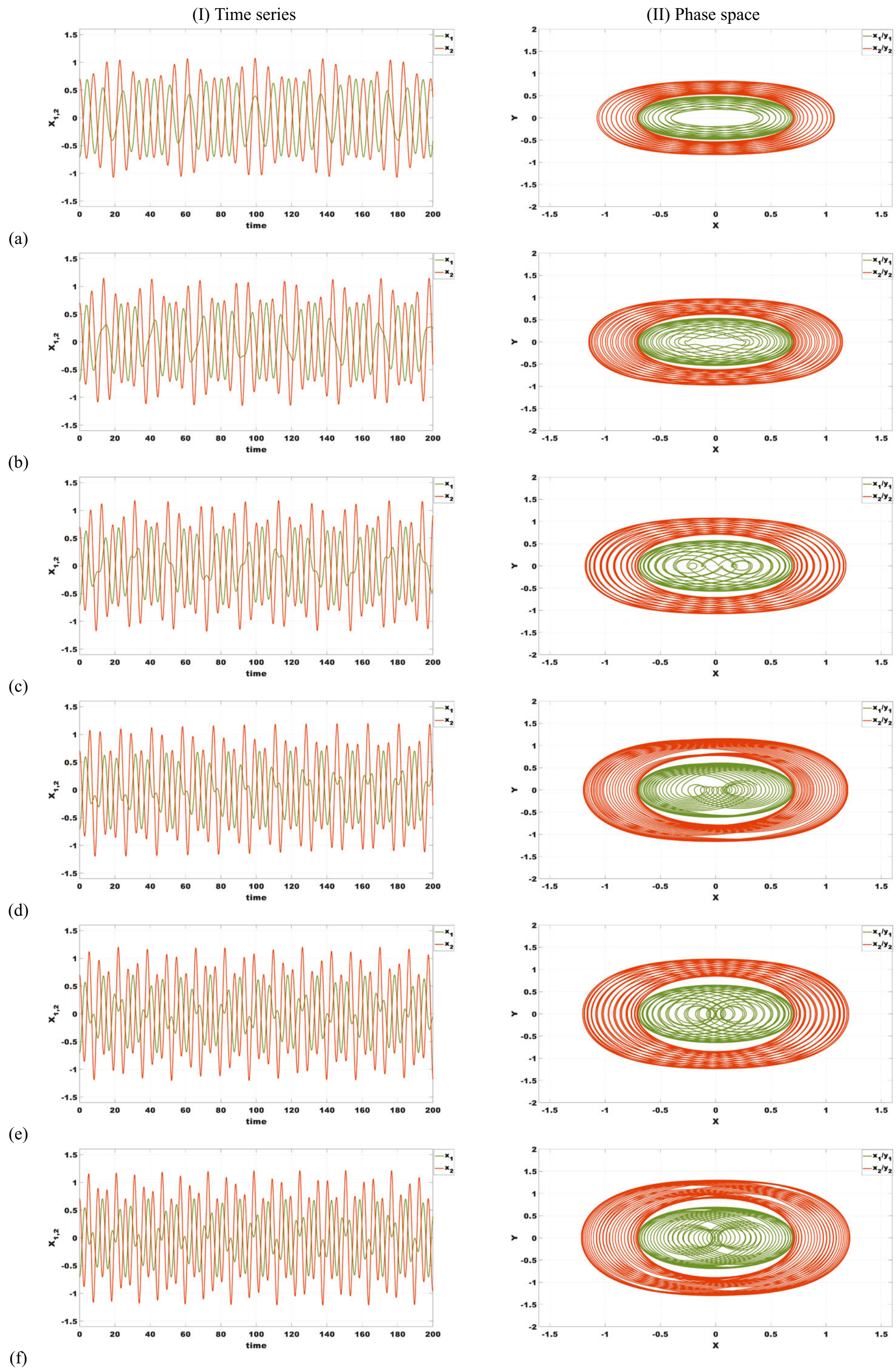

Fig. 4 Time series and phase space for single-well case with **a** $\beta = 0.05$, **b** $\beta = 0.10$, **c** $\beta = 0.15$, **d** $\beta = 0.20$, **e** $\beta = 0.25$ and **f** $\beta = 0.30$

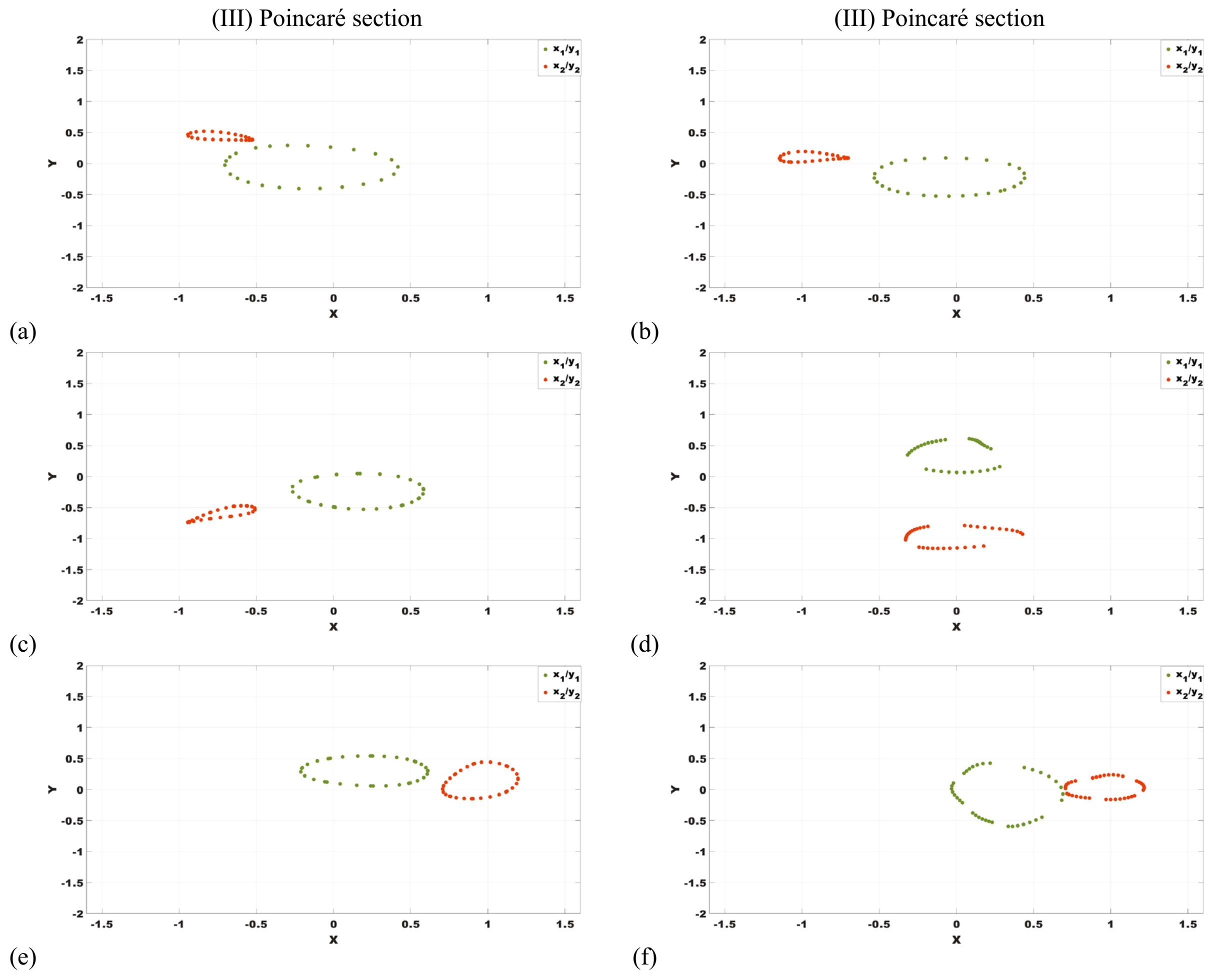

Fig. 5 Poincaré sections for single-well case with **a** $\beta = 0.05$, **b** $\beta = 0.10$, **c** $\beta = 0.15$, **d** $\beta = 0.20$, **e** $\beta = 0.25$ and **f** $\beta = 0.30$

4 Numerical results

In order to show the behavior of the system, we will firstly review the potential of the simple Duffing oscillator. It is well known that depending on the signs of the parameters Ω^2 and α, there are four possible behaviors [6]:

(a) If $\Omega^2 > 0$ and $\alpha < 0$, then the potential has a double-hump well with a local minimum at $x = 0$ and two maxima at $\pm\sqrt{\Omega^2/|\alpha|}$, as in Fig. 1a.
(b) If $\Omega^2 > 0$ and $\alpha > 0$, then the potential has a single well with a local minimum at $x = 0$, as in Fig. 1b.
(c) If $\Omega^2 < 0$ and $\alpha < 0$, then the potential has a single hump with a local maximum at $x = 0$, as in Fig. 1c.
(d) If $\Omega^2 < 0$ and $\alpha > 0$, then the potential has a double well with two minima at $\pm\sqrt{|\Omega^2|/\alpha}$ and a local maximum at $x = 0$, as in Fig. 1d.

It is clear that the double and simple hump cases do not present physical interest since the movement will be unbounded. Therefore, we will focus our attention on Duffing oscillator systems with double and simple well, where we will use the critical points $\pm\sqrt{|\Omega^2|/\alpha}$ as initial conditions. More specifically, we set the following initial data:

$$x_1(0) = x_3(0) = -\sqrt{\Omega^2/\alpha},$$
$$x_2(0) = \sqrt{\Omega^2/\alpha}, \qquad (11)$$
$$\dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_3(0) = 0,$$

with $\Omega^2 = 0.25$ and $\alpha = 0.5$.

The resulting spatiotemporal for x_1 and integral of motion planes of the system is shown in Fig. 2. In the first row, we can see the result for the single-well case where the behavior of x_1 is clearly stable as time

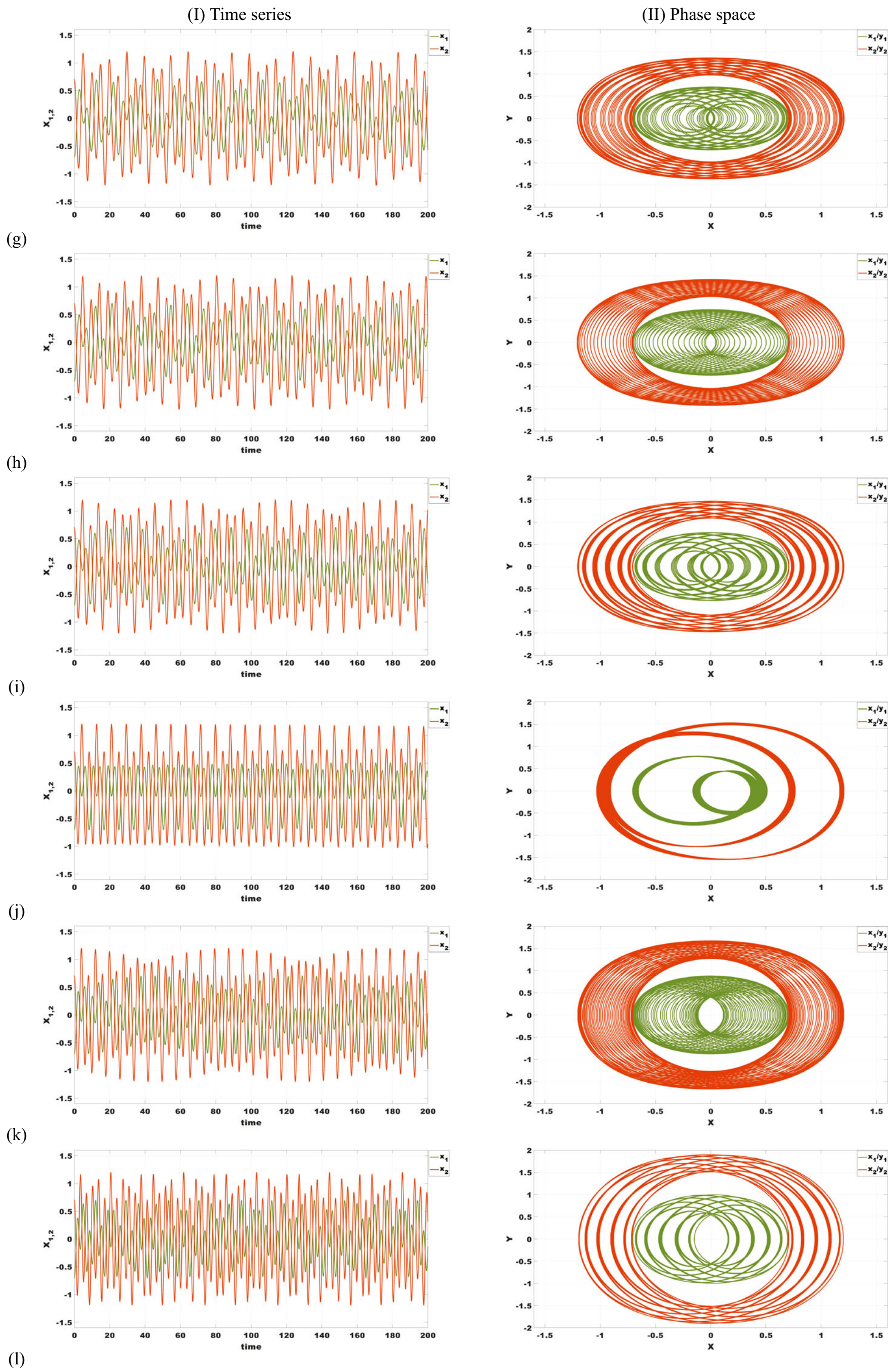

Fig. 6 Time series and phase space for single-well case with **g** $\beta = 0.35$, **h** $\beta = 0.40$, **i** $\beta = 0.45$, **j** $\beta = 0.56$, **k** $\beta = 0.65$ and **l** $\beta = 0.93$

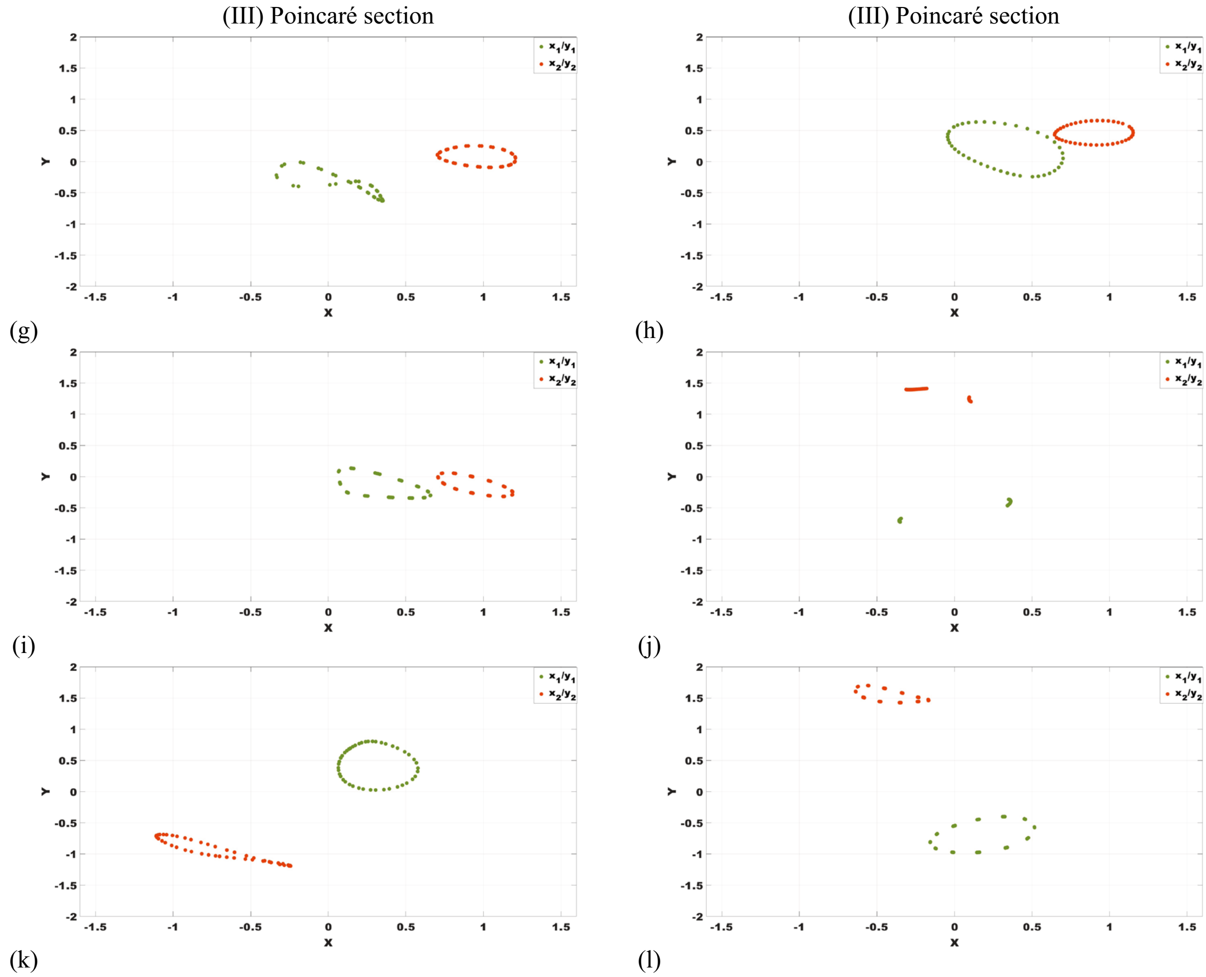

Fig. 7 Poincaré sections for single-well case with **g** $\beta = 0.35$, **h** $\beta = 0.40$, **i** $\beta = 0.45$, **j** $\beta = 0.56$, **k** $\beta = 0.65$ and **l** $\beta = 0.93$

progresses and the value of the coupling constant β increases (similar results are obtained for x_2 and x_3). Also, notice that the integral of motion increases with β, that is because the integral of motion (10) depends on the initial conditions (which are fixed), but it also depends on β, which increases from 0 to 1. The second row shows the result for the double well (also referred to as the bistable case). Here, we can see a different behavior of the oscillators, where a chaotic regime is presented for low values of parameter β (similar results are obtained for x_2 and x_3).

In order to analyze the nonlinear dynamics of the three-oscillator system, Fig. 3 shows the bifurcation diagrams and Lyapunov exponents for the simple and double-well cases. The local maximum values for x_1 and x_2 are plotted taking the coupling factor β as control parameter (note that x_3 is identical to x_1 by symmetry). Except for the range of amplitudes, the behaviors of x_1 and x_2 are similar. This is due to the fact that the variables are synchronized in phase. In the case of a simple well, the behavior is quasi-periodic, that is, there are two main frequencies. However, for low β values, we find few maxima that gradually increase as the coupling value increases. (Although for some specific β values, this number of maxima is considerably reduced.) The absence of chaotic behavior is confirmed with values of the Lyapunov exponent close to zero. For the case of a double well (as already mentioned), a chaotic behavior occurs at low values of the coupling parameter. More specifically, the value of the Lyapunov exponent associated with the oscillator x_1 has values clearly greater than zero when $0 < \beta < 0.3$. This

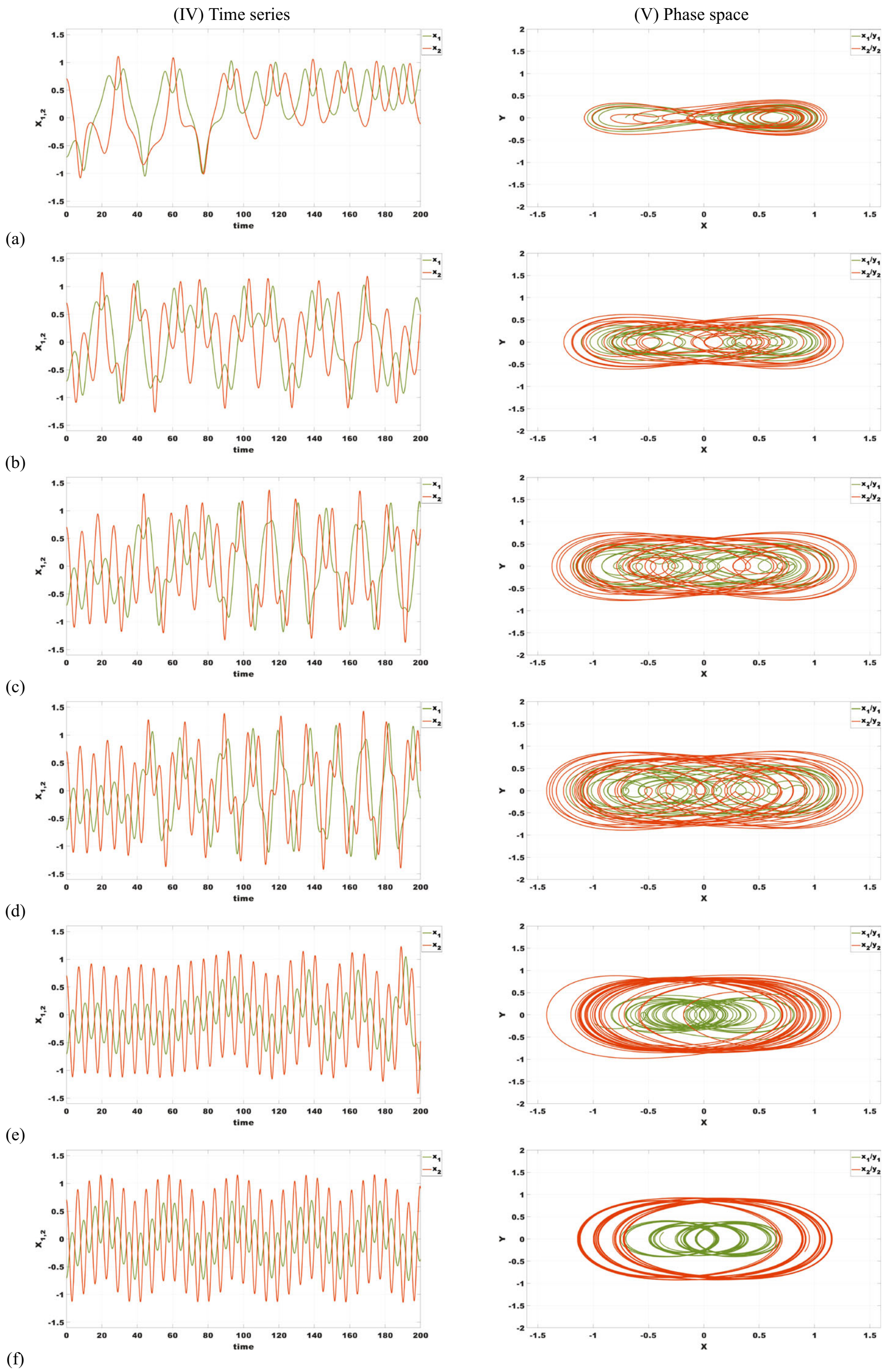

Fig. 8 Time series and phase space for double-well case with **a** $\beta = 0.05$, **b** $\beta = 0.10$, **c** $\beta = 0.15$, **d** $\beta = 0.20$, **e** $\beta = 0.25$ and **f** $\beta = 0.30$

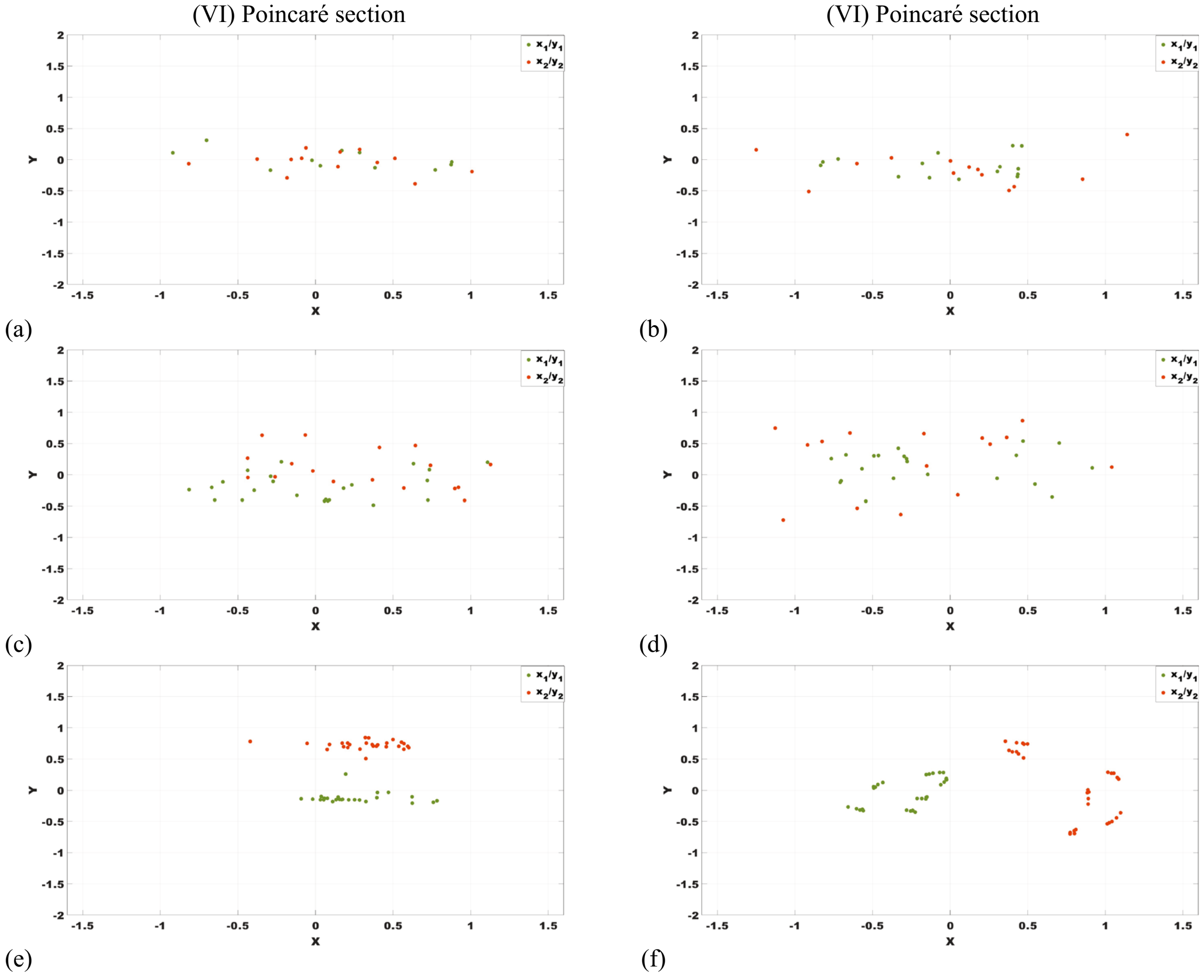

Fig. 9 Poincaré sections for double-well case with **a** $\beta = 0.05$, **b** $\beta = 0.10$, **c** $\beta = 0.15$, **d** $\beta = 0.20$, **e** $\beta = 0.25$ and **f** $\beta = 0.30$

behavior can be justified because there is a struggle for each of the oscillators to go from one well to another in a non-periodic way. The system gradually becomes quasi-periodic with few maxima for larger values of β, which increases as this parameter grows, that is, a similar behavior to that of the simple well is observed. This is once again shown by the values of the Lyapunov exponent close to zero.

To visualize in detail the individual behavior of the oscillators, Figs. 4, 5, 6, 7, 8, 9, 10, 11 (I,IV) show the time series, (II, V) show the phase space and (III,VI) show the Poincaré sections for different values of β between 0 and 1. These figures show the evolution of the dynamics in the system. In columns (I–III), which depict the behavior of the simple-well case, we can see the periodic, confined and practically exclusive movement between the oscillators x_1 and x_2. It can also be seen that for specific values of $\beta = 0.56, 0.93$, the region of motion narrows considerably. We can also observe that for the values included in the interval $0 < \beta < 0.5$ (see Figs. 4 (I,II) and 6 (I,II)), the variable x_1 remains in a quasi-periodic regime, while variable x_2 goes from quasi-periodic to homoclinic. Later, both regimes are synchronized in phase causing a similar behavior but with different amplitude, the period is the same and the shape of the attractor is similar. In turn, the Poincaré section shows a closed curve that indicates a quasi-periodic behavior where two frequencies are dominant. On the other hand, in columns (IV-VI) (the double-well case), the chaotic behavior of the oscillators is evident up to around a coupling parameter value $\beta = 0.25$, where the system roughly becomes periodic

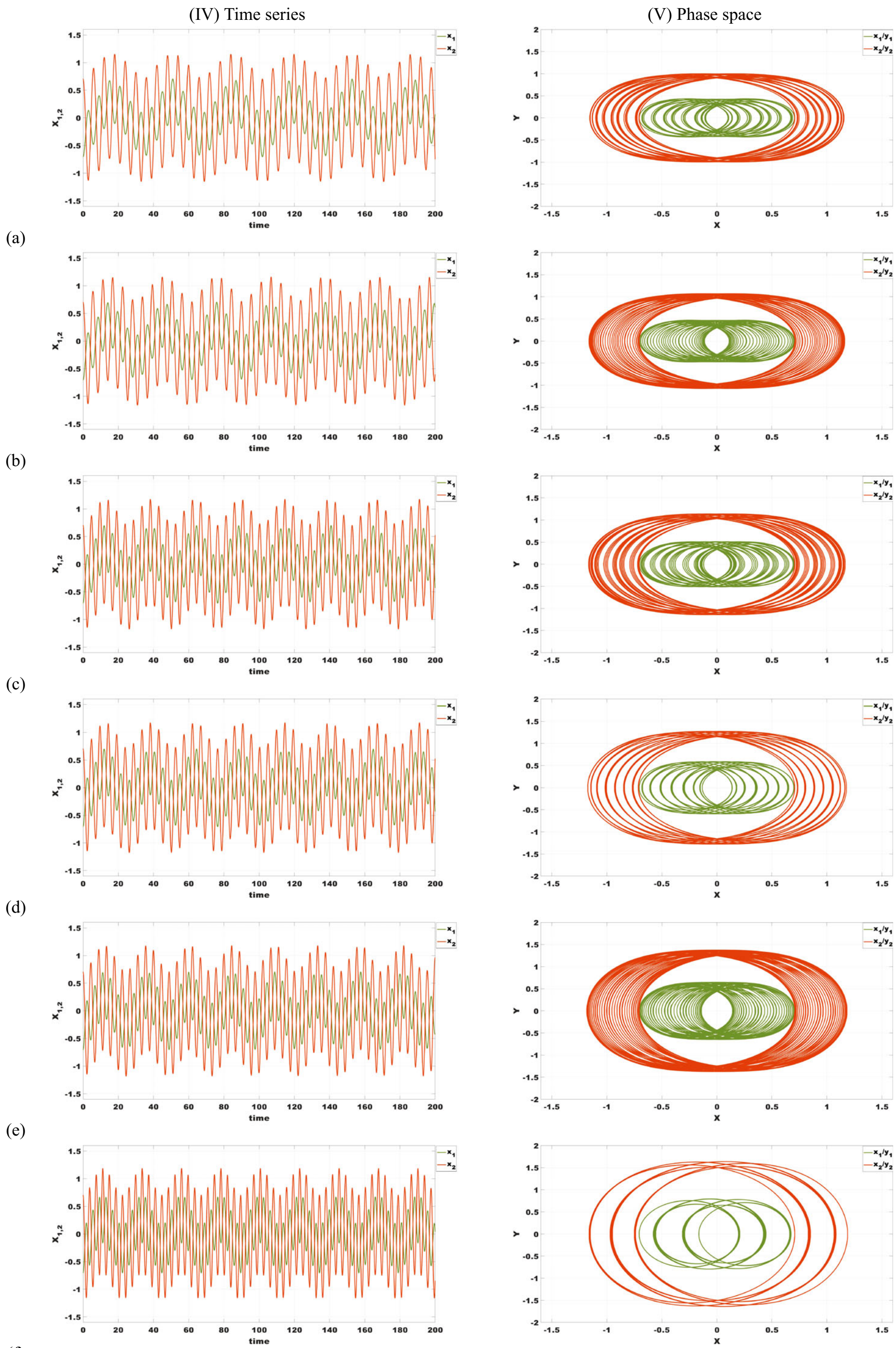

Fig. 10 Time series, phase space and Poincaré sections for double-well case with **a** $\beta = 0.05$, **b** $\beta = 0.10$, **c** $\beta = 0.15$, **d** $\beta = 0.20$, **e** $\beta = 0.25$ and **f** $\beta = 0.30$

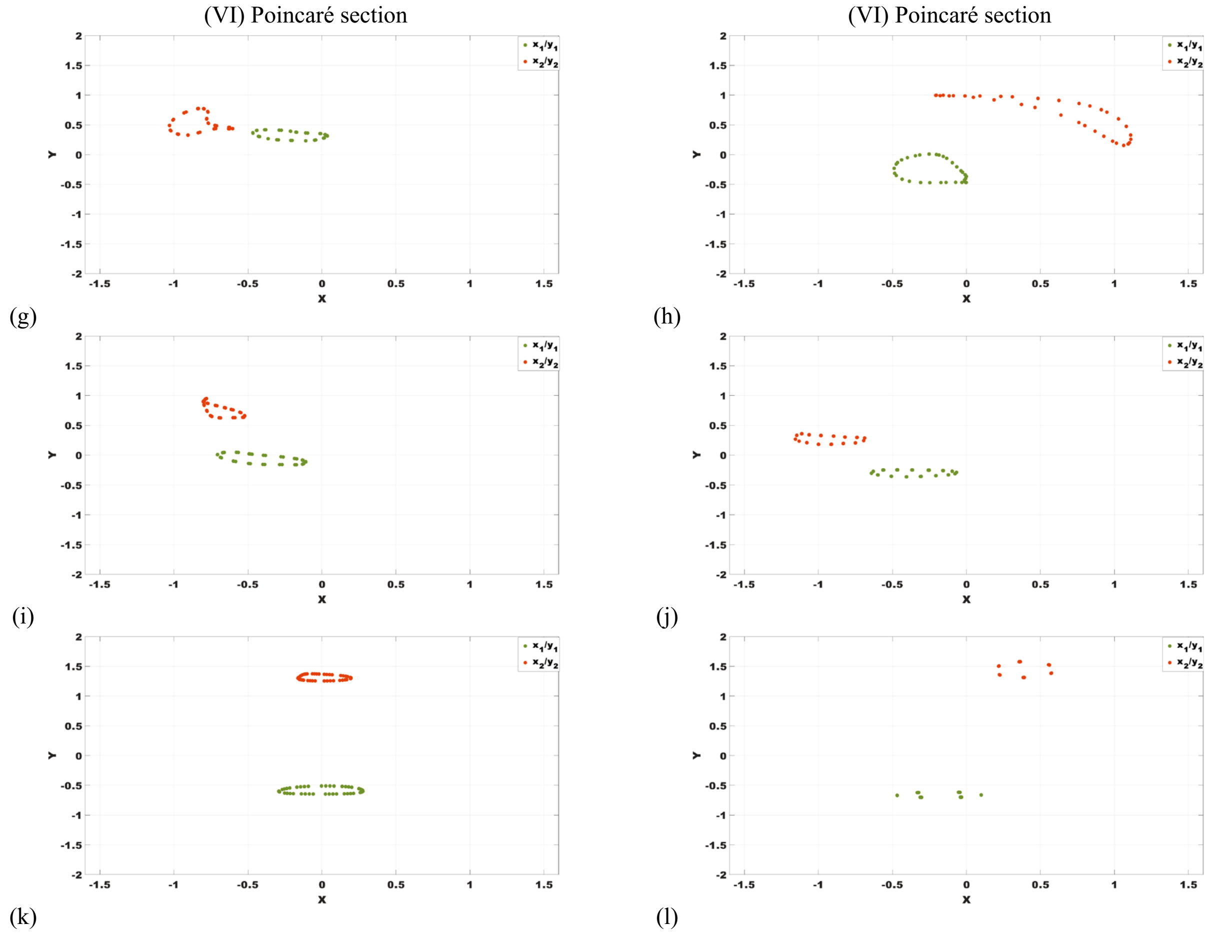

Fig. 11 Poincaré sections for double-well case with **g** $\beta = 0.35$, **h** $\beta = 0.40$, **i** $\beta = 0.45$, **j** $\beta = 0.56$, **k** $\beta = 0.65$ and **l** $\beta = 0.93$

and exclusive between oscillators, in a similar way to the previous case. Here, we can see that, for the specific value of $\beta = 0.93$, the region of motion narrows. More specifically, note that in Figs. 8 (II) and 10 (II), the phase space begins to fill without a clear shape of the attractor, while the Poincaré section (see Figs. 9 (III) and 11 (III)) shows a non-uniform distribution of points. When $\beta = 0.30$ (see Figs. 8 (I)(f), (II)(f) and 9 (III)(f)), the system changes its behavior and passes to quasi-periodic and homoclinic states which coexists for the rest of the values of the coupling constant (see Fig. 11 (I), (II) and (III)).

5 Conclusion

In this paper, we presented a system composed of three identical bidirectionally coupled Duffing oscillators. From a proposed Lagrangian, an integral of motion is obtained using the formalism of Noether's theorem. In order to study the dynamics of this system, we have shown the space-time and integral of motion planes in cases where the potential of the Duffing oscillator confines the motion. By means of bifurcation diagrams, Lyapunov exponents, time series, phase space and Poincaré sections, it was possible to visualize in detail the periodic and confined behavior of the oscillators and, more specifically, the manifestation of chaos for small values of the coupling parameter in the case of the double-well potential. This behavior confirms the results reported in [34], which may have significant implications in cardiology studies. We believe that the knowledge of an integral of motion for the case of a basic network of three nonlinear oscillators (such as the Duffing oscillators), where even chaotic behavior occurs, may be relevant in understanding complex networks where the chaotic behavior is of fundamental importance today.

Before closing this work, we would like to point out again that the literature in this area reports almost exclusively on the investigation of unidirectionally coupled systems of Duffing oscillators. In that context, some integrals of motion have been derived using various methodologies. Meanwhile, the study of systems with systems consisting of two bidirectionally coupled Duffing oscillators has been carried out assuming only the presence of weak couplings. After surveying thoroughly the current state of the art, the authors have found out that there are no reports on systems of three strongly coupled Duffing oscillators through bidirectional couplings. In that sense, the present work is one of the first successful efforts in the literature, in which non-trivial integrals of motion have been obtained for such complex systems. Obviously, one interesting problem to investigate in the future is to extend the results from the present manuscript to the case of a ring consisting of N oscillators (where $N \in \mathbb{N}$) with bidirectional coupling. Evidently, these results will help in studying the possible phase propagation and its chaotic behavior.

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Author contributions All authors contributed equally in this study. All authors read and approved the final manuscript

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Data availability The datasets generated during and/or analyzed during the current study are not publicly available but are available from the corresponding author (J.E.M.-D.) on reasonable request.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

References

1. Aldridge, J., Cleland, A.: Noise-enabled precision measurements of a Duffing nanomechanical resonator. Phys. Rev. Lett. **94**(15), 156403 (2005)
2. Arroyo-Almanza, D.A., Pisarchik, A.N., Ruiz-Oliveras, F.R.: Route to chaos in a ring of three unidirectionally-coupled semiconductor lasers. IEEE Photon. Technol. Lett. **24**(7), 605–607 (2012)
3. Barba-Franco, J., Gallegos, A., Jaimes-Reátegui, R., Muñoz-Maciel, J., Pisarchik, A.: Dynamics of coexisting rotating waves in unidirectional rings of bistable Duffing oscillators. Chaos Interdiscip. J. Nonlinear Sci. **33**(7), 073126 (2023)
4. Barba-Franco, J., Gallegos, A., Jaimes-Reátegui, R., Pisarchik, A.: Dynamics of a ring of three fractional-order Duffing oscillators. Chaos, Solitons & Fractals **155**, 111747 (2022)
5. Barba-Franco, J.J., Espinoza, P.B., Gallegos, A., Jaimes-Reategui, R., Macias-Diaz, J.E.: On the Lagrangians and potentials of a two coupled damped Duffing oscillators system and their application on three-node motif networks. Revista Mexicana de Física **66**(4), 440–445 (2020)
6. Barba-Franco, J.J., Gallegos, A., Jaimes-Reategui, R., Gerasimova, S.A., Pisarchik, A.N.: Dynamics of a ring of three unidirectionally coupled Duffing oscillators with time-dependent damping. Europhys. Lett. **134**(3), 30005 (2021)
7. Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., Hwang, D.U.: Complex networks: structure and dynamics. Phys. Rep. **424**(4–5), 175–308 (2006)
8. Boccaletti, S., Pisarchik, A.N., Del Genio, C.I., Amann, A.: Synchronization: from Coupled Systems to Complex Networks. Cambridge Univ Press, Cambridge (2014)
9. Borkowski, L., Perlikowski, P., Kapitaniak, T., Stefanski, A.: Experimental observation of three-frequency quasiperiodic solution in a ring of unidirectionally coupled oscillators. Phys. Rev. E **91**(6), 062906 (2015)
10. Borkowski, L., Stefanski, A.: Stability of the 3-torus solution in a ring of coupled Duffing oscillators. Eur. Phys. J. Special Top. **229**, 2249–2259 (2020)
11. Burgan, J.R., Feix, M.R., Fijalkow, E., Munier, A.: Solution of the multidimensional quantum harmonic oscillator with time-dependent frequencies through Fourier, Hermite and Wigner transforms. Phys. Lett. A **74**(1–2), 11–14 (1979)
12. Cervantes-López, E., Espinoza, P., Gallegos, A., Rosu, H.: Ermakov systems with multiplicative noise. Phys. A **401**, 141–147 (2014)
13. Cervantes-López, E., Espinoza, P., Gallegos, A., Rosu, H.: Ermakov–Ray–Reid systems with additive noise. Phys. A **439**, 244–47 (2015)
14. Cervantes-López, E., Espinoza, P., Gallegos, A., Rosu, H.C.: Additive and multiplicative noises acting simultaneously on Ermakov–Ray–Reid systems. Rev. Mex. Fis. **62**, 67–270 (2016)
15. Chen, J., Tepljakov, A., Petlenkov, E., Zhuang, B.: Boundary state and output feedbacks for underactuated systems of coupled time-fractional PDEs with different space-dependent diffusivity. Int. J. Syst. Sci. **51**(15), 2922–2942 (2020)
16. Chen, J., Zhuang, B.: Boundary control of coupled non-constant parameter systems of time fractional PDEs with

different-type boundary conditions. J. Syst. Sci. Complexity **36**(1), 273–293 (2023)
17. Chetaev, N.G.: The Stability of Motion. Pergamon Press, Oxford (1961)
18. Debnath, G., Zhou, T., Moss, F.: Remarks on stochastic resonance. Phys. Rev. A **39**(8), 4323 (1989)
19. Duffing, G.: Erzwungene Schwingungen bei veranderlicher Eigenfrequenz und ihre technische Bedeutung. 41–42. F. Vieweg & Sohn (1918)
20. Ermakov, V.: Second-order differential equations: conditions of complete integrability. Appl. Anal. Discrete Math. **2**, 123–45 (2008)
21. Ermentrout, G.: The behavior of rings of coupled oscillators. J. Math. Biol. **23**(1), 55–74 (1985)
22. Erturk, A., Inman, D.J.: Broadband piezoelectric power generation on high-energy orbits of the bistable Duffing oscillator with electromechanical coupling. J. Sound Vib. **330**(10), 2339–53 (2011)
23. Gallegos, A., Rosu, H.: Comment on demystifying the constancy of the Ermakov–Lewis invariant for a time-dependent oscillator. Mod. Phys. Lett. A **33**(24), 1875001 (2018)
24. Gallegos, A., Vargas-Rodríguez, H., Macías-Díaz, J.: Ermakov–Lewis invariants for a class of parametric anharmonic oscillators. Revista mexicana de física **63**(2), 162–165 (2017)
25. Gelfand, I.M., Silverman, R.A., et al.: Calculus of Variations. Courier Corporation (2000)
26. Haq, B., Naeem, I.: First integrals and analytical solutions of some dynamical systems. Nonlinear Dyn. **95**(3), 1747–1765 (2019)
27. Hu, N.Q., Wen, X.S.: The application of Duffing oscillator in characteristic signal detection of early fault. J. Sound Vib. **268**(5), 917–931 (2003)
28. Islam, S.O.B., Lughmani, W.A.: A connective framework for social collaborative robotic system. Machines **10**(11), 1086 (2022)
29. Jaimes-Reátegui, R., Castillo-Cruz, J., García-López, J., Huerta-Cuellar, G., Gallegos-Infante, L., Pisarchik, A.: Self-organization in network motifs of three bistable Duffing oscillators. Cybern. Phys. **9**(1), 31–40 (2020)
30. Jaros, P., Kapitaniak, T., Perlikowski, P.: Multistability in nonlinearly coupled ring of Duffing systems. Eur. Phys. J. Special Top. **225**, 2623–2634 (2016)
31. Jothimurugan, R., Thamilmaran, K., Rajasekar, S., Sanjuán, M.: Multiple resonance and anti-resonance in coupled Duffing oscillators. Nonlinear Dyn. **83**, 1803–1814 (2016)
32. Kamiyama, K., Endo, T.: Chaos of the propagating pulse wave in a ring of six-coupled bistable oscillators. Int. J. Bifurc. Chaos **22**(04), 1250091 (2012)
33. Kaushal, R., Korsch, H.: Dynamical Noether invariants for time-dependent nonlinear systems. J. Math. Phys. **22**(9), 1904–1908 (1981)
34. Keener, J.P.: Propagation and its failure in coupled systems of discrete excitable cells. SIAM J. Appl. Math. **47**(3), 556–572 (1987)
35. Korsch, H.: Dynamical invariants and time-dependent harmonic systems. Phys. Lett. A **74**, 294–296 (1979)
36. Kovacic, I.: Conservation laws of two coupled non-linear oscillators. Int. J. Non-Linear Mech. **41**(5), 751–760 (2006)
37. Kovacic, I., Brennan, M.J.: The Duffing Equation: Nonlinear Oscillators and Their Behaviour. Wiley (2011)
38. Kozłowski, J., Parlitz, U., Lauterborn, W.: Bifurcation analysis of two coupled periodically driven Duffing oscillators. Phys. Rev. E **51**(3), 1861 (1995)
39. Kudryashov, N.A.: The generalized Duffing oscillator. Commun. Nonlinear Sci. Numer. Simul. **93**, 105526 (2021)
40. Landau, L., Lifshiftz, E.: Mechanics, 2nd edn. Pergamon Press (1969)
41. LaSalle, J., Lefschetz, S.: Stability by Lyapunov's direct method. Academic Press, Cambridge (1961)
42. Leach, P.G., Andriopoulos, K.: The Ermakov equation: a commentary. Appl. Anal. Discrete Math. **2**, 146–157 (2008)
43. Lei, Y., Guan, F.: Disorder induced order in an array of chaotic Duffing oscillators. Int. J. Modern Phys. C **23**(10), 1250071 (2012)
44. Lenci, S.: Exact solutions for coupled Duffing oscillators. Mech. Syst. Signal Process. **165**, 108299 (2022)
45. Leung, A.Y.T., Chui, S.: Non-linear vibration of coupled Duffing oscillators by an improved incremental harmonic balance method. J. Sound Vib. **181**(4), 619–633 (1995)
46. Lewis, J.H.: Class of exact invariants for classical and quantum time-dependent harmonic oscillators. J. Math. Phys. **9**(11), 1976–86 (1968)
47. Lutzky, M.: Noether's theorem and the time-dependent harmonic oscillator. Phys. Lett. A **68**, 3–4 (1978)
48. Macías-Díaz, J., Urenda-Cázares, E., Gallegos, A.: Driven damped nth-power anharmonic oscillators with time-dependent coefficients and their integrals of motion. Results Phys. **25**, 104169 (2021)
49. Milo, R., Shen-Orr, S., Itzkovitz, S., Kashtan, N., Chklovskii, D., Alon, U.: Network motifs: simple building blocks of complex networks. Science **298**(5594), 824–827 (2002)
50. Musielak, D., Musielak, Z., Benner, J.: Chaos and routes to chaos in coupled Duffing oscillators with multiple degrees of freedom. Chaos Solitons Fractals **24**(4), 907–922 (2005)
51. Naz, R., Naeem, I., Mahomed, F.M.: First integrals for two linearly coupled nonlinear Duffing oscillators. Math. Probl. Eng. **2011**, 831647 (2011)
52. Nefdt, R.M.: Biolinguistics and biological systems: a complex systems analysis of language. Biol. Philosop. **38**(2), 12 (2023)
53. Nijmeijer, H., Berghuis, H.: On Lyapunov control of the Duffing equation. IEEE Trans. Circuits Syst. I **42**(8), 473–7 (1995)
54. Padmanabhan, T.: Demystifying the constancy of the Ermakov–Lewis invariant for a time-dependent oscillator. Mod. Phys. Lett. A **33**(07n08), 18300005 (2018)
55. Parlitz, U., Lauterborn, W.: Superstructure in the bifurcation set of the Duffing equation $x + dx + x + x^3 = f \cos(\omega t)$. Phys. Lett. A **107**(8), 351–5 (1985)
56. Perlikowski, P., Yanchuk, S., Wolfrum, M., Stefanski, A., Mosiolek, P., Kapitaniak, T.: Routes to complex dynamics in a ring of unidirectionally coupled systems. Chaos Interdiscip. J. Nonlinear Sci. **20**(1), 013111 (2010)
57. Pulikkottil, J.J., Lakshminarayan, A., Srivastava, S.C., Kieler, M.F., Bäcker, A., Tomsovic, S.: Quantum coherence controls the nature of equilibration and thermalization in coupled chaotic systems. Phys. Rev. E **107**(2), 024124 (2023)

58. Ray, J., Reid, J.: More exact invariants for the time-dependent harmonic oscillator. Phys. Lett. A **71**, 317–18 (1979)
59. Ray, J.R., Reid, J.L.: Invariants for forced time-dependent oscillators and generalizations. Phys. Rev. A **26**(2), 1042 (1982)
60. Reategui, R.J., Pisarchik, A.N.: Control of on-off intermittency by slow parametric modulation. Phys. Rev. E **69**(6), 067203 (2004)
61. Sharkawy, A.N., Koustoumpardis, P.N., Aspragathos, N.: Human-robot collisions detection for safe human-robot interaction using one multi-input-output neural network. Soft. Comput. **24**, 6687–6719 (2020)
62. Shimizu, K., Komuro, M., Endo, T.: Onset of the propagating pulse wave in a ring of coupled bistable oscillators. Nonlinear Theory Appl. IEICE **2**(1), 139–151 (2011)
63. Soriano, J.: Neuronal cultures: exploring biophysics, complex systems, and medicine in a dish. Biophysica **3**(1), 181–202 (2023)
64. Tchakui, M.V., Woafo, P.: Dynamics of three unidirectionally coupled autonomous Duffing oscillators and application to inchworm piezoelectric motors: Effects of the coupling coefficient and delay. Chaos Interdiscip. J. Nonlinear Sci. **26**(11), 113108 (2016)
65. Turing, A.M.: The chemical basis of morphogenesis. Bull. Math. Biol. **52**, 153–197 (1990)
66. Urenda-Cázares, E., Espinoza, P., Gallegos, A., Jaimes-Reátegui, R., Macías-Díaz, J.E., Vargas-Rodríguez, H.: The noisy Pais-Uhlenbeck oscillator. J. Math. Chem. **57**, 1314–1329 (2018)
67. Urenda-Cázares, E., Gallegos, A., Jaimes-Reátegui, R.: Effects of multiplicative noise on the Duffing oscillator with variable coefficients and its integral of motion. Int. J. Mod. Phys. C **31**(07), 2050095 (2020)
68. Urenda-Cázares, E., Gallegos, A., Macías-Díaz, J.E., Vargas-Rodríguez, H.: An integral of motion for the damped cubic-quintic Duffing oscillator with variable coefficients. Commun. Nonlinear Sci. Numer. Simul. **78**, 104860 (2019)
69. Van der Sande, G., Soriano, M.C., Fischer, I., Mirasso, C.R.: Dynamics, correlation scaling, and synchronization behavior in rings of delay-coupled oscillators. Phys. Rev. E **77**(5), 055202 (2008)
70. Wong, J.: On the generalized Emden–Fowler equation. SIAM Rev. **17**(2), 339–60 (2003)
71. Zhang, J., Zhang, L., An, X., Luo, H., Yao, K.E.: Adaptive coupled synchronization among three coupled chaos systems and its application to secure communications. EURASIP J. Wirel. Commun. Netw. **2016**, 1–15 (2016)
72. Zhang, Y., Hu, G., Cerdeira, H.A.: How does a periodic rotating wave emerge from high-dimensional chaos in a ring of coupled chaotic oscillators? Phys. Rev. E **64**(3), 037203 (2001)